

Abstract Algebra II

Problem Set 1: Solvability

Exam style questions are marked with (*).

Commutant

Question 1. Show the commutant is a normal subgroup, i.e. $K(G) \triangleleft G$.

Question 2. (*)

- Prove that the quotient of a group by its commutant is commutative.
- If $N \triangleleft G$ prove that G/N is commutative if and only if $K(G) < N$.

Conclude that the larger the index of $K(G)$ in G , the “less abelian” G is.

Question 3. Let G be a group with $K(G) = G$. Is G necessarily abelian? If so, provide a proof. If not, provide a counterexample.

Question 4. (*) Find the commutant of the group of symmetric transformations of a regular tetrahedron.

Question 5. (*) Find the commutant of the group of rotations of the cube and the group of rotations of the regular octahedron. Find the relationship between the two groups.

Soluble groups

Question 6. (*) Prove that subgroups of soluble groups are soluble.

Question 7. Let $\varphi : G \rightarrow F$ be a surjective homomorphism.

- Suppose that G is soluble. Prove that F is soluble.
- Provide an example of a soluble F and non soluble G .

Question 8. Find the chain of commutants of S_4 . (You may assume any results from class.)

Question 9. Argue whether S_n is soluble for $n > 4$.

Question 10. • Let $N \trianglelefteq G$. If G is soluble prove that G/N is soluble.

- Let $N \trianglelefteq G$. If N and G/N are both soluble prove that G is soluble.